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Backreaction: directions of progress

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Abstract: Homogeneous and isotropic cosmological models with ordinary matter and gravity predict slower expansion and shorter distances than observed. It is possible that this failure is due the known breakdown of homogeneity and isotropy related to structure formation, rather than new fundamental physics. We review this backreaction conjecture, concentrating on topics on which there has been progress as well as open issues.

Contents

1. Introduction

1.1 Three choices for a factor of two

The predictions of homogeneous and isotropic models (with linear perturbations) with ordinary matter and gravity are off by a factor of about two compared to observations in the late universe. Ordinary matter here means that it has non-negative pressure and ordinary gravity refers to the four-dimensional Einstein-Hilbert action. The simplest such model, which contains particles of the Standard Model of particle physics and cold dark matter in a spatially flat universe, underpredicts the distance to the last scattering surface at redshift 1090 by a factor of 1.4–1.7, for a fixed Hubble parameter today (assuming a power-law spectrum of primordial perturbations) [1]. The expansion rate today is wrong by a factor of 2 if we keep

the matter density fixed, $\Omega_{\rm m0} \equiv 8\pi G_{\rm N}\rho_{\rm m0}/(3H_0^2) \approx 0.25$ [2], or a factor of 1.2–1.5 if we keep the age of the universe fixed instead, $H_0 t_0 \approx 0.8$–1 as opposed to $H_0 t_0 = 2/3$ [3, 4]. This factor of two disagreement means that at least one of the three assumptions is wrong. Either there is exotic matter with negative pressure, general relativity does not hold on cosmological scales, or the homogeneous and isotropic approximation is not valid at late times.

Mathematically, the simplest possibility is to retain the homogeneous and isotropic approximation and introduce vacuum energy or a cosmological constant, which are examples of exotic matter and modified gravity, respectively. Quantum field theory predicts that there is an energy density associated with the vacuum state, so this possibility is theoretically on very solid ground (unlike most other exotic matter or modified gravity proposals). This ΛCDM model agrees well with most observations, though there appear to be discrepancies in the distribution of matter on large scales [5–8]. Its main problem is generally considered to be the fact that the value of the vacuum energy required to explain the observations is very small compared to known particle physics scales, $\rho_{\rm vac} \approx (2.3\ \text{meV})^4$. However, there is no prediction for the vacuum energy, only arguments based on naturalness. It can even be argued that the meV scale is quite natural from the point of view of electroweak physics, as follows. In the axiomatic approach to quantum field theory in curved space [9], the vacuum energy of a free scalar field vanishes in the limit of zero mass. Let us assume that the same is true for gauge fields and fermions, and that the Higgs is a composite, so that there are no fundamental scalar fields. At high energies, when the electroweak symmetry is unbroken, the vacuum energy would be zero, while a non-zero value would be generated in the electroweak phase transition. Naively, we would expect this energy to be of the order of the electroweak scale. However, it is expected that in an interacting theory vacuum expectation values depend non-analytically on the coupling constants [9]. We can make the simple estimate $\rho_{\rm vac} = v^4 e^{-\frac{1}{g^2}} = v^4 e^{-\frac{1}{\alpha}} \approx (1.3\times10^{-15} v)^4 \approx (0.3\ \text{meV})^4$, where $v = 246$ GeV is the Higgs vacuum expectation value and g^2 is the coupling, for which we have simply put $\alpha = 1/137$. Substituting for the scale v the sum of particle masses and taking into account numerical prefactors would change the vacuum energy by factors of order unity, and it is of course exponentially sensitive to factors of 4π and weak mixing angles in the exponent (and sensitive to the scale at which the coupling is evaluated). Like arguments in favor of unified scale vacuum energy, this is little more than inspired numerology, but it shows that it is not implausible to get the right scale out from the quantum field theory of electroweak physics.

The vacuum energy density required to explain the observations is considered problematic not only because of its smallness compared to known fundamental scales, but also because it has to be close to the matter density today. Another formulation of this coincidence problem is that the vacuum energy would have become impor-

tant only at late times, when the universe is about ten billion years old. It seems serendipitous that we should be witnessing a special and brief dynamical phase in the evolution of the universe, when the universe is undergoing a transition from matter domination to vacuum energy domination. However, the coincidence problem contradicts neither observation nor any known theoretical law, so at most it provides a motivation to search for alternatives.

In contrast, a concrete problem with all homogeneous and isotropic models is that the real universe is far from homogeneity and isotropy at late times. Indeed, homogeneous and isotropic models with ordinary matter and gravity agree well with observations of early times, and the factor of two disagreement arises in the late universe when deviations from homogeneity and isotropy become significant.

Before concluding that the introduction of vacuum energy or more complicated new physics is needed, it is necessary to check the validity of the homogeneous and isotropic approximation. The effect of deviations from homogeneity and isotropy on average quantities (in particular, the average expansion rate) is called backreaction [10–13]. Physically, the simplest possibility is that the factor two discrepancy would be explained by the known breakdown of the homogeneous and isotropic approximation related to to structure formation, without any new fundamental physics [14–19]. This may be called *the backreaction conjecture*.

The formation of non-linear structures does in fact lead to deviations in the local expansion rate which are of the order of the observed discrepancy, and the process has a preferred time of about ten billion years. The key issue is how local deviations add up and cancel in the observed signal, which involves integrals over large scales. Note that the success of the homogeneous and isotropic model with vacuum energy shows that the observations can be explained simply by increasing the expansion rate (and correspondingly making distances longer), since that is the only cosmological effect of vacuum energy. Furthermore, there is no evidence from local physics for exotic matter or modified gravity, all of the indications involve integrals over large scales. The situation is rather different from that of dark matter, for which there are several independent lines of evidence [20], including from local physics. Also, since most observations probe distances, which involve the expansion rate only via an integral (for exceptions, see [21, 22]), the expansion history can in fact have significant deviations from the homogeneous and isotropic vacuum model while still fitting the data.

The problem is well-defined. Given the known particle content plus a model of dark matter and starting from a nearly homogeneous and isotropic state at early times, with a given Gaussian spectrum of perturbations, how do the matter distribution and geometry evolve in general relativity? In particular, we want to find how null geodesics are affected, since most cosmological observations consist of measurements of photons. The difficulty arises from the complexity of non-linear evolution in general relativity. The problem of finding a tractable approximation scheme is com-

plicated by the fact that it is not immediately obvious what are the relevant physical effects which have to be included. At the moment it is not yet clear whether backreaction is quantitatively important or not, but there has been important progress in understanding the phenomenon.

In section 2 we discuss the basics of how structures affect the average expansion rate. In section 3 we illustrate the issue with a simple toy model which has correct qualitative features, and then present a semi-realistic model where we can estimate magnitude of the effect. In section 4 we briefly discuss light propagation and its relation to the average expansion rate. In section 5 we discuss the role of the Newtonian limit of general relativity and linear perturbation theory. We conclude in section 6 with a summary.

2. The expansion rate

2.1 Statistical and exact symmetry

In cosmology, the evolution of the universe is usually described with the homogeneous and isotropic Friedmann-Robertson-Walker (FRW) models, with the justification that the universe appears to be homogeneous and isotropic on large scales. However, it is important to distinguish between *exact* and *statistical* homogeneity and isotropy. Exact homogeneity and isotropy means that the space has a local symmetry: all points and all directions are equivalent. The FRW models are exactly homogeneous and isotropic. Statistical homogeneity and isotropy simply means that if we consider a box anywhere in the universe, the mean quantities in the box do not depend on its location, orientation or size, provided that it is larger than the homogeneity scale. (See [8] for a more detailed discussion of statistical homogeneity and isotropy, and also the issue of self-averaging.)

The early universe is nearly exactly homogeneous and isotropic, in two ways. First, the amplitude of the perturbations around homogeneity and isotropy is small. Second, the distribution of the perturbations is statistically homogeneous and isotropic. At late times, when density perturbations become non-linear, the universe is no longer locally near homogeneity and isotropy, and there are deviations of order unity in quantities such as the local expansion rate. However, the distribution of the non-linear regions remains statistically homogeneous and isotropic on large scales. It has been argued that the homogeneity scale would have been detected [23], but the result is disputed [8]; in any case the homogeneity scale is not less than 100 Mpc. We assume that the universe is indeed statistically homogeneous and isotropic, with a homogeneity scale much smaller than the Hubble scale. We are interested in the effects of the structures that are known to exist, not speculative structures such as Gpc-scale voids.

Due to the statistical symmetry, the average expansion rate evaluated inside each box is equal (up to statistical fluctuations), but this does not mean that it would be the same as in a completely smooth spacetime, because there are structures in the boxes. We can say that time evolution and averaging do not commute: if we smooth a clumpy distribution and calculate the time evolution of the smooth quantities with the Einstein equation, the result is not the same as if we evolved the full clumpy distribution and took the average at the end. Put simply, FRW models describe universes which are exactly homogeneous and isotropic, not universes which are only statistically homogeneous and isotropic. The effect of clumpiness on the average was first discussed in detail by George Ellis in 1983 under the name *fitting problem* [24]. Clumpiness affects the expansion of the universe, the way light propagates in the universe and the relationship between the two. Let us first discuss the expansion rate.

2.2 The local expansion rate

We consider a universe where the energy density of matter dominates over pressure, anisotropic stress and energy flux everywhere. In other words, the matter can be considered a pressureless ideal fluid, or dust. We assume that the relation between the matter and the geometry is given by the Einstein equation:

$$G_{\alpha\beta} = 8\pi G_{\rm N} T_{\alpha\beta} = 8\pi G_{\rm N} \rho u_\alpha u_\beta \ , \tag{2.1}$$

where $G_{\alpha\beta}$ is the Einstein tensor, $G_{\rm N}$ is Newton's constant, $T_{\alpha\beta}$ is the energy–momentum tensor, ρ is the energy density and u^α is the velocity of observers comoving with the dust. In the real universe, the matter cannot locally be treated as dust everywhere, but the deviations are unlikely to be relevant for quantities integrated over large scales, which is what enters into the observations. For treatment of non-dust matter, see [25, 26].

The evolution and constraint equations can be written elegantly in terms of the gradient of u_α and the electric and magnetic components of the Weyl tensor [27, 28],

$$\nabla_\beta u_\alpha = \frac{1}{3} h_{\alpha\beta}\theta + \sigma_{\alpha\beta} + \omega_{\alpha\beta} \ , \tag{2.2}$$

where $h_{\alpha\beta}$ projects orthogonally to u^α. The trace $\theta \equiv \nabla_\alpha u^\alpha$ is the volume expansion rate, the traceless symmetric part $\sigma_{\alpha\beta}$ is the shear tensor and the antisymmetric part $\omega_{\alpha\beta}$ is the vorticity tensor. For an infinitesimal fluid element, θ indicates how its volume changes in time, keeping the shape and the orientation fixed, while shear changes the shape and vorticity changes the orientation. In the FRW case, the volume expansion rate is just $3H$, where H is the Hubble parameter.

The equations can be be decomposed into scalar, vector and tensor parts with respect to the spatial directions orthogonal to u^α. We need only the scalar parts (we

omit a scalar equation related to the vorticity),

$$\begin{aligned}
\dot{\theta} + \frac{1}{3}\theta^2 &= -4\pi G_{\rm N}\rho - 2\sigma^2 + 2\omega^2 && (2.3)\\
\frac{1}{3}\theta^2 &= 8\pi G_{\rm N}\rho - \frac{1}{2}{}^{(3)}R + \sigma^2 - \omega^2 && (2.4)\\
\dot{\rho} + \theta\rho &= 0 \ , && (2.5)
\end{aligned}$$

where a dot stands for derivative with respect to proper time t measured by observers comoving with the dust, $\sigma^2 \equiv \frac{1}{2}\sigma^{\alpha\beta}\sigma_{\alpha\beta} \geq 0$ and $\omega^2 \equiv \frac{1}{2}\omega^{\alpha\beta}\omega_{\alpha\beta} \geq 0$ are the shear scalar and the vorticity scalar, respectively. In the irrotational case, ${}^{(3)}R$ is the spatial curvature of the hypersurface which is orthogonal to u^α; see [29] for the definition in the case of non-vanishing vorticity.

Equation (2.5) simply shows that the energy density is proportional to the inverse of the volume, in other words that mass is conserved. The second equation (2.4) is the local equivalent of the Friedmann equation, and it relates the expansion rate to the energy density, spatial curvature, shear and vorticity. The equation (2.3) gives the local acceleration. Let us assume that the fluid is irrotational, i.e. that the vorticity is zero. (See [26] for the case with vorticity.) As vorticity contributes positively to the acceleration, putting it to zero gives a lower bound. In this case, the local acceleration is always negative, or at most zero. This is just an expression of the fact that gravity is attractive for matter which satisfies the strong energy condition.

Cosmological distance observations imply that the expansion rate has accelerated if we assume that the FRW relation between distance and the expansion rate holds. Deviations from homogeneity and isotropy change this relationship, so this conclusion does not necessarily hold in the real universe; we discuss this in section 4. (Based on direct measurements of the expansion rate, we can only say that there has been less deceleration, not that the expansion has accelerated.) We can distinguish between apparent and actual acceleration. Apparent acceleration means that when cosmological observations are interpreted assuming that the universe is well described by a FRW model, the expansion rate given by the FRW scale factor has accelerated. Actual acceleration means that the real volume expansion rate has really increased in time. It is easy to understand how a different relationship between the expansion rate and distance might lead to apparent acceleration, but it is possible for inhomogeneities to lead to actual acceleration as well, if we consider the average expansion rate relevant for cosmological observations.

2.3 The average expansion rate

When discussing averages, the first question concerns the choice of the hypersurface on which the average is taken. We choose the hypersurface orthogonal to u^α, which is also the hypersurface of constant proper time t measured by the observers. (Discussion of this choice is postponed to section 4.) The spatial average of a scalar quantity

f is its integral over the hypersurface, with the correct volume element, divided by the volume:

$$\langle f\rangle(t) \equiv \frac{\int d^3x \sqrt{^{(3)}g(t,\bar{x})}\, f(t,\bar{x})}{\int d^3x \sqrt{^{(3)}g(t,\bar{x})}} \ , \tag{2.6}$$

where $^{(3)}g$ is the determinant of the metric on the hypersurface of constant proper time t.

Averaging (2.3)–(2.5), we obtain the Buchert equations [30]

$$3\frac{\ddot{a}}{a} = -4\pi G_{\rm N}\langle\rho\rangle + \mathcal{Q} \tag{2.7}$$

$$3\frac{\dot{a}^2}{a^2} = 8\pi G_{\rm N}\langle\rho\rangle - \frac{1}{2}\langle^{(3)}R\rangle - \frac{1}{2}\mathcal{Q} \tag{2.8}$$

$$\partial_t\langle\rho\rangle + 3\frac{\dot{a}}{a}\langle\rho\rangle = 0 \ , \tag{2.9}$$

where the backreaction variable $\mathcal{Q}$ contains the effect of inhomogeneity and anisotropy,

$$\mathcal{Q} \equiv \frac{2}{3}\left(\langle\theta^2\rangle - \langle\theta\rangle^2\right) - 2\langle\sigma^2\rangle \ , \tag{2.10}$$

and the scale factor $a(t)$ is defined so that the volume of the spatial hypersurface is proportional to $a(t)^3$,

$$a(t) \equiv \left(\frac{\int d^3x \sqrt{^{(3)}g(t,\bar{x})}}{\int d^3x \sqrt{^{(3)}g(t_0,\bar{x})}}\right)^{\frac{1}{3}} \ , \tag{2.11}$$

where a has been normalised to unity at time t_0, which we take to be today. As θ gives the expansion rate of the volume, this definition of a is equivalent to $3\dot{a}/a \equiv \langle\theta\rangle$. We also use the notation $H \equiv \dot{a}/a$.

The Buchert equations (2.7)–(2.9) have a slightly different physical interpretation than the FRW equations due to the different meaning of the scale factor. In FRW models, the scale factor is a component of the metric, and indicates how the space evolves locally. In the present context, $a(t)$ does not describe local behaviour, and it is not part of the metric. It simply gives the total volume of a region.

Mathematically, the Buchert equations differ from the FRW equations by the presence of the backreaction variable $\mathcal{Q}$ and the related feature that the average spatial curvature can have non-trivial evolution. In the FRW case, $\mathcal{Q} = 0$ and $\langle^{(3)}R\rangle \propto a^{-2}$. This evolution of the spatial curvature follows from the integrability condition between (2.7) and (2.8), and it is also an independently known feature of any homogeneous and isotropic model [31]. Note that $\mathcal{Q}$ can vanish even when the universe is locally far from FRW. In other words, the FRW equations may give a correct description of the average evolution even if they are completely wrong for

the local dynamics. (By derivation, the FRW equations are meant to describe local evolution.)

In general, $\mathcal{Q}$ is non-zero, and it expresses the non-commutativity of time evolution and averaging. The backreaction variable $\mathcal{Q}$ has two parts: the second term in (2.10) is the average of the shear scalar, which is also present in the local equations (2.3)–(2.5). It is always negative (unless the spacetime is FRW, in which case it is zero), and acts to decelerate the expansion. In contrast, the first term in (2.10), the variance of the expansion rate, has no local counterpart. It may be called emergent in the sense that it is purely a property of the average system. The variance is always positive (unless the expansion is homogeneous, in which case it is zero). If the variance is sufficiently large compared to the shear and the energy density, the average expansion rate accelerates according to (2.7), even though (2.3) shows that the local expansion rate decelerates everywhere.

3. Modelling backreaction

3.1 A two-region toy model

It may seem paradoxical that the average expansion rate accelerates even though the local expansion rate decelerates everywhere at all times. So let us first consider a simple toy model to understand the physical meaning before moving on to a semi-realistic model of the universe. We give the punchline right away. In an inhomogeneous space, different regions expand at different rates. Regions with faster expansion rate increase their volume more rapidly, by definition. Therefore the fraction of volume in faster expanding regions rises, so the average expansion rate can rise. Whether the average expansion rate actually does rise depends on how rapidly the fraction of fast regions grows relative to the rate at which their expansion rate decelerates.

In the early universe, the distribution of perturbations of the density, and thus of the expansion rate, is very smooth, with only small local variations. In a simplified picture, overdense regions slow down more as their density contrast grows, and eventually they turn around and collapse to form stable structures. Underdense regions become ever emptier, and their deceleration decreases. Regions thus become more differentiated and the variance of the expansion rate grows.

We can illustrate this with a simple toy model where there are two spherically symmetric regions, one underdense and one overdense [11, 32]. We consider the regions to be Newtonian, so their evolution is given by the spherical collapse model and the underdense equivalent, i.e. they expand like dust FRW universes with negative and positive spatial curvature, respectively. We denote the scale factors of the underdense and the overdense region by a_1 and a_2, respectively. We take the underdense region, which models a cosmological void, to be completely empty, so it expands

like $a_1 \propto t$. The evolution of the overdense region, which models the formation of a structure such as a cluster, is given by $a_2 \propto 1 - \cos\phi$, $t \propto \phi - \sin\phi$, where the parameter ϕ is called the development angle. The value $\phi = 0$ corresponds to the big bang singularity, from which the overdense region expands until $\phi = \pi$, when it turns around and starts collapsing. The region shrinks to zero size at $\phi = 2\pi$. In studies of structure formation, the collapse is usually taken to stabilise at $\phi = 3\pi/2$ due to vorticity and velocity dispersion, and we also follow the evolution only up to that point. The total volume is $a^3 = a_1^3 + a_2^3$. The average expansion rate and acceleration are

$$H = \frac{a_1^3}{a_1^3 + a_2^3} H_1 + \frac{a_2^3}{a_1^3 + a_2^3} H_2 \equiv v_1 H_1 + v_2 H_2 \tag{3.1}$$

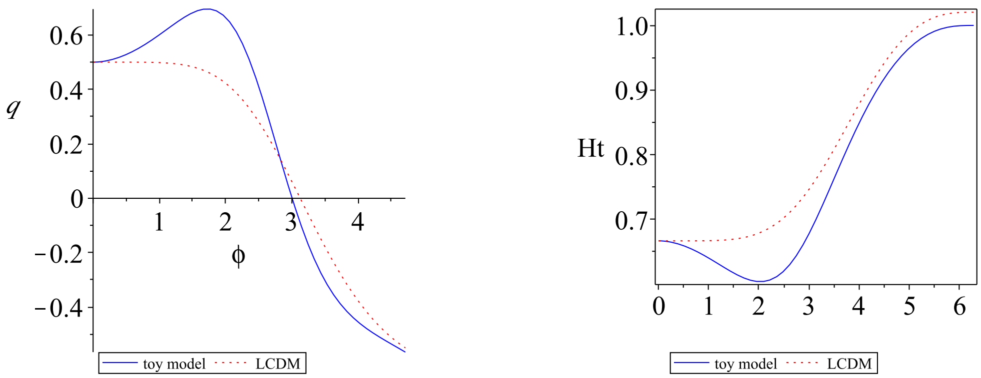
$$\frac{\ddot{a}}{a} = v_1 \frac{\ddot{a}_1}{a_1} + v_2 \frac{\ddot{a}_2}{a_2} + 2 v_1 v_2 (H_1 - H_2)^2 \ . \tag{3.2}$$

The average expansion rate is the volume-weighted average of the expansion rates H_1 and H_2, as one would expect. It is therefore bounded from above by the fastest local expansion rate. However, from the fact that both H_1 and H_2 decrease it does not follow that their weighted average would decrease, or that the average expansion rate would decelerate. This is illustrated by the acceleration equation (3.2). The first two terms are the volume-weighted average, and because the regions decelerate (or at most have zero acceleration, in the completely empty case), it is negative. However, there is an additional term related to the difference between the two expansion rates, which is always positive (as long as the regions have non-zero volume and different expansion rates). This term arises because a time derivative of (3.1) operates not only on H_1 and H_2, but also on v_1 and v_2. In terms of the general acceleration equation (2.7), the first two terms in (3.2) come from the average density, and the last term is (one third of) the backreaction variable $\mathcal{Q}$.

The toy model has one free parameter, the relative size of the two regions at some time. For illustration purposes, we fix this by setting the deceleration parameter $q \equiv -\ddot{a}/a/H^2$ at $\phi = 3\pi/2$ to the value of the spatially flat ΛCDM FRW model with $\Omega_\Lambda = 0.7$. In figure 1 (a) we plot q as a function of the development angle ϕ. We also show the ΛCDM model for comparison.

The ΛCDM model starts matter-dominated, with $q = 1/2$. As vacuum energy becomes important, the model decelerates less and then crosses over to acceleration. Asymptotically, q approaches -1 from above as the Hubble parameter approaches a constant value. The backreaction model also starts with the FRW matter-dominated behaviour, then the expansion slows down more, before q turns around and the expansion decelerates less and eventually accelerates: in fact the acceleration is stronger than in the ΛCDM model.

The acceleration is not due to regions speeding up locally, but due to the slower region becoming less represented in the average. First the overdense region brings



(a) **(b)**

Figure 1: The evolution of the toy model as a function of the development angle ϕ. (a): The deceleration parameter q in the toy model (blue, solid) and in the ΛCDM model (red, dash-dot). (b): The Hubble parameter multiplied by time, Ht, in the toy model (blue, solid) and in the ΛCDM model (red, dash-dot).

down the expansion rate, but its fraction of the volume falls because of the slower expansion, so eventually the underdense region takes over and the average expansion rate rises. This is particularly easy to understand after the overdense region has started collapsing at $\phi = \pi$. Then the contribution $v_2 H_2$ of the overdense region to (3.1) is negative, and its magnitude shrinks rapidly as v_2 decreases, so it is transparent that the expansion rate increases. Note that while there is an upper bound on the expansion rate, there is no lower bound on the collapse rate. Therefore, the acceleration can be arbitrarily rapid, and q can even reach minus infinity in a finite time. (This simply means that the negative expansion rate of the collapsing region cancels becomes equal to the positive expansion rate of the exanding region, so H vanishes in the denominator of q.) This is in contrast to FRW models, where $q \geq -1$ unless the null energy condition (or the modified gravity equivalent) is violated. After the overdense region stops being important, the expansion rate will be given by the underdense region alone, and the expansion will again decelerate. Acceleration is a transient phenomenon associated with the volume becoming dominated by the underdense region.

Figure 1 (b) shows the Hubble parameter multiplied by time as a function of the development angle ϕ. This contains the same information as figure 1 (a), but plotted in terms of the first derivative of the scale factor instead of the second derivative. In the ΛCDM model, Ht starts from 2/3 in the matter-dominated era and increases monotonically without bound as H approaches a constant. In the toy model, Ht falls as the overdense region slows down, then rises as the underdense region takes over, approaching unity from below. The Hubble parameter in the toy model is smaller than in the ΛCDM model at all times. Because H is bounded from above by the fastest local expansion rate, Ht cannot exceed unity. This bound also holds

in realistic models: as long as matter can be treated as dust and vorticity can be neglected, we have $Ht \leq 1$ at all times [33], in contrast to FRW models with exotic matter or modified gravity. This is a prediction of backreaction. (For discussion of vorticity and non-dust terms in the energy-momentum tensor, see [25, 26].)

Whether the expansion accelerates depends on how rapidly the faster expanding regions catch up with the slower ones, roughly speaking by how steeply the Ht curve rises. This is why the variance contributes positively to acceleration: the larger the variance, the bigger the difference between fast and slow regions, and the more rapidly the fast regions take over.

3.2 A statistical semi-realistic model

The toy model shows how acceleration due to inhomogeneities can occur and makes transparent what this means physically. Acceleration has also been demonstrated with the exact spherically symmetric dust solution, the Lemaître-Tolman-Bondi model [34–36]. So there is no ambiguity: accelerated average expansion due to inhomogeneities is possible. The question is whether the distribution of structures in the universe is such that this mechanism is realised. The statement that faster expanding regions increase their volume more rapidly makes it sound as if there would necessarily be less deceleration (if not acceleration) than in the FRW case. For a set of isolated regions, this is true: eventually, the volume will be dominated by the fastest region. However, the characteristic feature of structures in the real universe is their hierarchical buildup. Smaller structures become incorporated into larger ones, and rapidly expanding voids can be extinguished by collapsing clouds.

The non-linear evolution of structures is too complex to follow exactly. However, because the universe is statistically homogeneous and isotropic, statistical properties are enough to evaluate the average expansion rate. The average expansion rate is determined if we know which fraction of the universe is in which state of expansion or collapse. Instead of trying to find a solution for the metric and calculating the quantities of interest from it, we can consider an ensemble of regions from which we can determine the average expansion rate without having to consider the global metric. We now discuss a semi-realistic model which does this by extending the two fixed regions of the toy model to a continuous distribution of regions which evolves in time [37, 38].

The starting point is the spatially flat matter-dominated FRW model with a linear Gaussian field of density fluctuations. Structure formation, even though complicated, is a deterministic process. Therefore any statistical quantity at late times is determined by the initial distribution processed by gravity. For a Gaussian distribution, the power spectrum contains all statistical information. So even in the completely non-linear regime, the average expansion rate follows from the power spectrum. The problem is formulating a tractable model for propagating the structures given by the initial power spectrum into the non-linear regime with gravity.

One approach, proposed in [39], is to identify structures at late times with spherical peaks in the original linear density field, smoothed on an appropriate scale. The number density of peaks as a function of the smoothing scale and peak height can be determined analytically in terms of the power spectrum. In the original application, the correspondence between peaks and structures was assumed to hold only for very non-linear overdense structures: all peaks exceeding a certain density threshold were identified with stabilised structures. Here the idea is a bit different: spherical peaks of any density are identified with structures having the same linear density contrast. Troughs are identified with spherical voids in the same way. (As the distribution is Gaussian, the statistics of peaks and troughs are identical.) We keep the smoothing threshold fixed such that $\sigma(t, R) = 1$, where σ is the root mean square linear density contrast, t is time and R is the smoothing scale. Non-linear structures form at $\sigma \approx 1$, so R corresponds to the size of the typical largest structures, and grows in time. The smoothing is just a simplified treatment of the complex stabilisation and evolution of structures in the process of hierarchical structure formation.

Since the peaks are spherical and isolated, and they are individually assumed to be in the Newtonian regime, their expansion rate is the same as that of a dust FRW universe with the same density, as in the toy model. The volume which is neither in peaks nor in troughs is taken to expand like the spatially flat matter-dominated FRW model.

The peak number density as a function of time is determined by the power spectrum, which consists of two parts: the primordial power spectrum, determined in the early universe by inflation or some other process, and the transfer function, which describes the evolution between the primordial era and the time when the modes enter the non-linear regime. The transfer function $T(k)$ simply multiplies the amplitude of the primordial modes. We take a scale-invariant primordial spectrum with the observed amplitude; small variations from scale-invariance have little effect. For the transfer function, we assume that dark matter is cold, and we consider two different approximations in order to show the uncertainty in the calculation. The BBKS transfer function [39] is a fit to numerical calculations (we take a baryon fraction of 0.2), and the BDG form introduced in [40] is a simple analytically tractable function with the correct qualitative features. The transfer functions are shown in figure 2 as a function of $k/k_{\rm eq}$, the wavenumber divided by the matter-radiation equality scale. Modes with $k > k_{\rm eq}$ enter the horizon during radiation domination, so their amplitude is damped. The sooner they enter, the more they are damped before the universe becomes matter-dominated, so there is a damping tail, which falls approximately like k^{-2}. Modes with $k < k_{\rm eq}$ enter during the matter-dominated era and retain their original amplitude. For modes with $k \sim k_{\rm eq}$, the transfer function interpolates between these two regimes. In the BBKS transfer function, the transition is centered around $k_{\rm eq}$ and is rather gradual, while in the BDG case the transition happens a bit earlier and is more rapid. Even the more realistic BBKS transfer

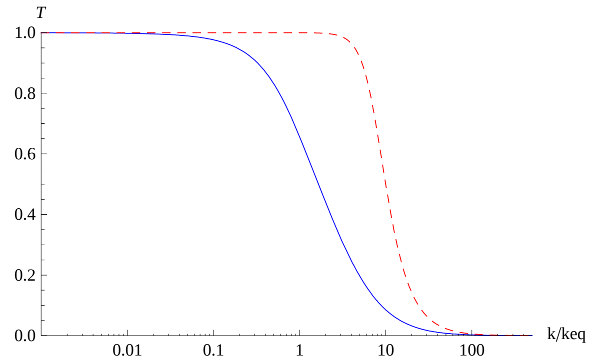


Figure 2: The BBKS (blue, solid) and BDG (red, dashed) transfer functions as a function of $k/k_{\rm eq}$.

function has an error of 20–30% compared to Boltzmann codes.

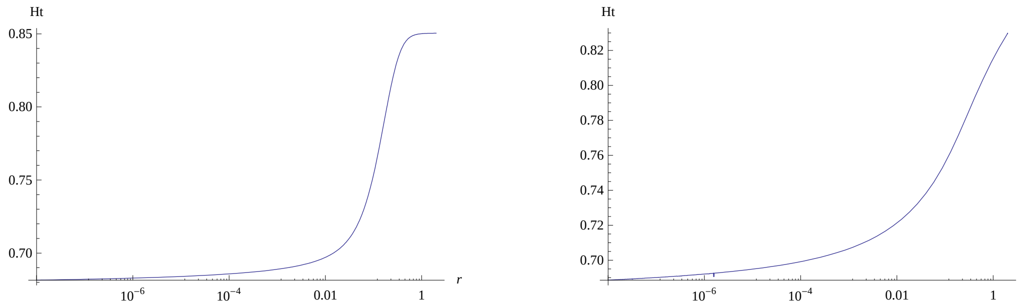
We have

$$H(t) = \int_{-\infty}^{\infty} \mathrm{d}\delta \, v_\delta(t) H_\delta(t) \ , \tag{3.3}$$

where $v_\delta \mathrm{d}\delta$ is the fraction of volume in regions with linear density contrast δ and expansion rate $H_\delta(t)$. The correspondence between δ and H_δ is given by the spherical evolution model (i.e. FRW evolution), and the distribution of regions $v_\delta(t)$ is given by the peak statistics, which is determined by the power spectrum of the Gaussian density field. With the transfer function fixed, there are no free parameters: the expansion history $H(t)$ given by (3.3) is completely determined. Since the primordial spectrum is scale-invariant and the smoothing and peak identification process does not introduce a scale, features in the expansion rate as a function of time can only come from the turnover at the matter-radiation equality scale in the transfer function.

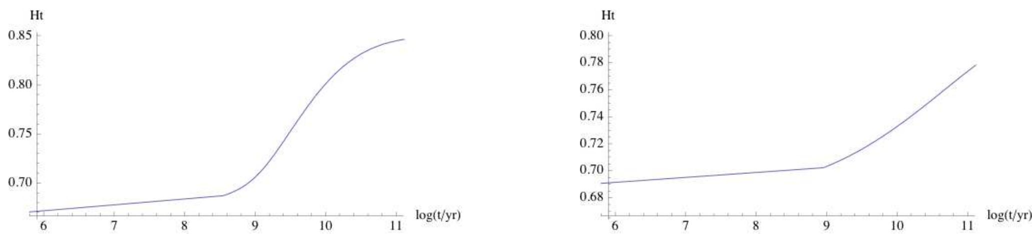
In figure 3 we show Ht as a function of $r \equiv k_{\rm eq} R$, the smoothing scale relative to the matter-radiation equality scale. Essentially, the coordinate r is time as measured by the size of the largest generation of structures. We have $Ht \approx 2/3$ at early times, as the fraction of volume in non-linear structures is small. As time goes on, deeper non-linear structures form, and they take up a larger fraction of the volume. The expansion rate grows (relative to the FRW value) slowly, until there is rapid rise and saturation, roughly at the scale of matter-radiation equality. It is clear that after $r = 1$, when the perturbations which correspond to the matter-radiation equality scale collapse, Ht must settle to a constant, since the transfer function is nearly unity, and there is no scale in the system anymore.

The matter-radiation equality scale is $k_{\rm eq}^{-1} \approx 13.7 \omega_{\rm m}^{-1}$ Mpc ≈ 100 Mpc, using the value $\omega_{\rm m} = 0.14$ [1]. Observationally, $\sigma(t, R) \approx 1$ today on scales somewhat smaller than 8 h^{-1}Mpc, so $R_0 \approx 10$ Mpc. Therefore the present day happens to be located around $r = 0.1$ in the plots – right in the transition region. Note that nothing related to present day has been used as input in the calculation.



(a) **(b)**

Figure 3: The expansion rate Ht as a function of $r = k_{\rm eq}R$ for (a) the BDG transfer function and (b) the BBKS transfer function.



(a) **(b)**

Figure 4: The expansion rate Ht as a function of time for (a) the BDG transfer function and (b) the BBKS transfer function.

It is instructive to view Ht also as function of time as measured in years. In figure 4, the horizontal axis is $\log_{10}(t/\mathrm{yr})$. For the BDG transfer function, Ht has the FRW value at one million years, and it grows very slowly until it rises at about a billion years, and then saturates to a value somewhat larger than 0.8 at some tens of billions of years. For the more realistic BBKS transfer function, the behaviour is qualitatively the same, but the transition is slower and the final value of Ht is smaller. The slope of the Ht curve is less steep as a function of time than as a function of r, because the size of structures grows more slowly at late times. When plotting Ht as function of the smoothing scale, the comparison scale is $k_{\rm eq}$, whereas here it is the time of matter-radiation equality, $t_{\rm eq}$. Now the amplitude of the primordial perturbations also enters. The timescale follows from the shape of the transfer function. Perturbations which entered the horizon at matter radiation equality reach non-linearity at $t \approx A^{-3/2} t_{\rm eq} \approx$ 100 Gyr, where $A = 3 \times 10^{-5}$ is the primordial amplitude and the matter-radiation equality time is $t_{\rm eq} \approx 1000 \omega_{\rm m}^{-2}$ years $\approx 50\,000$ years for $\omega_{\rm m} = 0.14$. This is when the expansion rate saturates, and it enters the transition region somewhat earlier.

As noted in section 3.1, whether or not the expansion accelerates is a quantitative question related to the slope of the Ht curve. In the present case, while the expansion rate increases relative to the FRW value, the change is not sufficiently rapid for the

expansion to accelerate, there is just less deceleration. This is related to the fact that, unlike in the toy model, the overdense regions play almost no role, and the evolution of Ht can be understood in terms of the underdense voids. At early times, voids take up only a small part of the volume, and Ht rises smoothly as their volume fraction increases.

Nevertheless, it is encouraging that the model gives a change of the right order of magnitude in Ht, 15–25%, and that the timescale for the change comes out right. The model involves many approximations, such as treating structures as spherical, using an approximate transfer function, having an artificial split between the peaks/troughs and the smooth space, not taking into account that the Gaussian symmetry between the overdense and underdense regions is broken in the non-linear regime (in the present treatment, they have equal mass in at all times) and treating the structures as isolated even for small density contrasts and high peak number densities. It is clear the model cannot be trusted beyond an order of magnitude. It is also possible that a more careful statistical treatment would reveal cancellations that significantly change backreaction from this approximate estimate.

In order to obtain a more drastic change in Ht, the expansion rate should have extra deceleration due to overdense regions before the voids take over, as in the toy model. (This effect is present in the model, but it is too small to be visible in figures (3) and (4).) If the expansion were to slow down more before the voids take over, the variance and the change in the expansion rate would be larger. The magnitude of the change of Ht is easy to understand: if the universe were completely dominated by totally empty voids, we would have $Ht = 1$. Since not all of the volume is taken up by voids and they are not totally empty, Ht is somewhat smaller than unity. As noted in the case of the toy model, $Ht < 1$ is a prediction of backreaction [33], assuming that matter can be treated as dust and rotation can be neglected [26]. The constraint $Ht < 1$ also means that proposals for implementing primordial inflation using backreaction [41] are unfeasible (aside from other problems, such as generating the spectrum of primordial perturbations).

The change in the expansion can also be viewed in terms of the deceleration parameter q (see [37] for the plots). From (2.7) and (2.10) we have $q = \frac{1}{2}\Omega_{\rm m} - 2(\langle\theta^2\rangle - \langle\theta\rangle^2)/\langle\theta\rangle^2 + 6\langle\sigma^2\rangle/\langle\theta\rangle^2$. We can obtain a conservative lower bound on this parameter by taking into account $\langle\sigma^2\rangle > 0$ and $\langle\theta^2\rangle < (3t^{-1})^2$, where the latter inequality follows from the fact that the local expansion rate cannot be higher than $3t^{-1}$. This gives $q > \frac{1}{2}\Omega_{\rm m} - 2[(Ht)^{-2} - 1]$. For a realistic distribution of structures, the value of q is likely to be much above this bound. For the values $\Omega_{\rm m0} = 0.3$ and $H_0t_0 = 0.8\ldots1$ we have $q_0 > -0.98\ldots0.15$. There is a tension between obtaining a large value of Ht and a very negative value of q simultaneously. The physical reason is that in order to have Ht close to unity, a large fraction of the volume has to be in regions which are nearly empty. This in turn means that the variance, and hence acceleration, is smaller. In terms of the spatial curvature, we see from

(2.7) and (2.8) that for $\Omega_{\rm m0} = 0.3$ and $q_0 = -0.55$, we have $\langle {}^{(3)}R\rangle_0 = -6.3H_0^2$, or $\Omega_{R0} \equiv -\langle {}^{(3)}R\rangle_0/(6H_0^2) = 1.05 > 1$. The spatial curvature is large, because much of the volume is occupied by very underdense regions. We should note two caveats with regard to q_0. First, while we have model-independent measurements of the Hubble parameter today, determinations of the deceleration parameter from the data are very dependent on the assumed parametrisation of the expansion history [42]. For backreaction, we would expect extra deceleration before the acceleration, and the expansion will return to deceleration as the voids take over. This sort of evolution is excluded by construction in most parametrisations of the expansion history. The data, however, does not exclude the possibility that the expansion could have already have gone from acceleration back to deceleration. There have been arguments that the observations would in fact slightly prefer deceleration today [43] and extra deceleration before acceleration [44], though such trends are not statistically significant in the present data. Another caveat is that the deceleration parameter is determined from distance observations, assuming that the relation between distance and expansion rate is the one given by the FRW metric. However, clumpiness changes this relation, a topic we now turn to.

4. Light propagation

4.1 The choice of hypersurface

Historically, studies of the average expansion rate and light propagation have been somewhat disconnected. In perturbative light propagation calculations, the change of the average evolution has often been neglected, while studies of the average expansion rate have usually not made the connection to observations of light. However, the primary quantities are the observable redshift and distance, and averages are useful only insofar as they give an approximate description of what is observed [26,45]. As we discuss below, it is the requirement that the average expansion rate describes light propagation which fixes the hypersurface of averaging. The physically relevant averaging hypersurface cannot be determined on abstract mathematical grounds. This is a crucial feature, given that the averages depend on the choice of hypersurface. Note that the averaging hypersurface is a physical choice, and should not be confused with choice of coordinates nor choice of gauge [19,46,47]. The derivation of the Buchert equations (2.7)–(2.9) is entirely covariant, and the result is uniquely defined in terms of measurable quantities. It does not depend on coordinates (indeed, it is not necessary to specify the coordinate system). Because the treatment is non-perturbative and does not refer to a background, there is also no question of gauge choice (which refers to a mapping between the real spacetime and a fictitious background).

It has been argued that the procedure of averaging only scalar quantities is somehow incomplete, and various proposals have been put forth for averaging tensors.

The macroscopic gravity formalism [48], for example, extends general relativity so that one can map the physical manifold onto another manifold, which is in some sense an average of the real one. However, the issue at hand is the effect of deviations from spatial homogeneity and isotropy in the fixed spacetime geometry which describes the real universe, not quantities calculated in some other spacetime. (See section 4.1 of [26].) Averages and ensembles can be useful for describing cosmological observations which probe large scales because of the statistical homogeneity and isotropy of the universe. However, it has to be demonstrated that they really describe observational quantities.

Almost all cosmological observations are made along the lightcone, measuring the redshift, the angular diameter distance (or equivalently the luminosity distance) and other quantities related to bundles of light rays. In a general spacetime, these quantities are not determined solely by expansion, and certainly not by the average expansion rate along spacelike slices of simultaneity. However, in a statistically homogeneous and isotropic universe where the distribution evolves slowly, the average expansion rate does determine the leading behaviour of the redshift and the distance [26, 45]. Considering the real observables also fixes the choice of averaging hypersurface. We now sketch the argument for this.

4.2 The redshift

In a general dust spacetime, the redshift is given by (see [26] for the non-dust case)

$$1 + z = \exp\left(\int_{\eta}^{\eta_0} \mathrm{d}\eta \left[\frac{1}{3}\theta + \sigma_{\alpha\beta}e^{\alpha}e^{\beta}\right]\right) \ , \tag{4.1}$$

where η is defined by $\partial/\partial\eta \equiv (u^{\alpha} + e^{\alpha})\partial_{\alpha}$, and e^{α} is the spatial direction of the null geodesic. If there are no preferred directions and the change in the distribution is slow compared to the time it takes for a light ray to pass through a homogeneity scale sized region, the integral over $\sigma_{\alpha\beta}e^{\alpha}e^{\beta}$ is suppressed. In the real universe, if the homogeneity scale is around 100 Mpc, then it is indeed much smaller than the timescale for the change in the distribution, which is given by the Hubble scale $H_0^{-1} = 3000h^{-1}$Mpc. In the early universe, structure formation is less advanced, so the homogeneity scale is even smaller relative to the Hubble scale further down the null geodesic. The direction e^{α} changes slowly for typical light rays [26], whereas the dust shear is correlated with the shape and orientation of structures and changes on the length scale of those structures. If there are no preferred directions, over large scales structures are oriented in all directions equally, so $\sigma_{\alpha\beta}$ should contribute via its trace, which is zero. Therefore the integral over $\sigma_{\alpha\beta}e^{\alpha}e^{\beta}$ should vanish, up to statistical fluctuations and corrections from correlations between $\sigma_{\alpha\beta}$ and e^{α} and evolution of the distribution. We can split the local expansion rate as $\theta = \langle\theta\rangle + \Delta\theta$, where $\Delta\theta$ is the local deviation from the average, and similarly argue that the integral of $\Delta\theta$ is suppressed relative to the contribution of the average expansion rate. This

suppression of the dependence on direction also explains how the small anisotropy of the cosmic microwave background (CMB) is not in contradiction with order unity perturbations in the geometry [49].

Here the choice of hypersurface is important. For the cancellations to occur, the averaging has to be done on the hypersurface of statistical homogeneity and isotropy. (In addition, the evolution of the distribution from one hypersurface to another has to be slow compared to the homogeneity scale.) This defines the hypersurface of averaging. In section 2.3 we took the average on the hypersurface of constant proper time of observers comoving with the matter. Since the evolution of structures is governed by the proper time, one can argue that this is close to the hypersurface of statistical homogeneity and isotropy [11, 37, 45]. However, these hypersurfaces will not be exactly the same, and in the realistic case when the observer velocity is not irrotational, the hypersurface of constant proper time is not orthogonal to the observer velocity. The details are thus more complicated, but non-relativistic changes in the velocity field which defines the hypersurface of averaging lead only to small changes in the averages, as long as the distribution is statistically homogeneous and isotropic, and the averaging scale is at least as large as the homogeneity scale [26].

Given that $\langle\theta\rangle = 3\dot{a}/a$, we obtain $1+z \approx a(t)^{-1}$, the same relation between expansion and redshift as in the FRW case. This result depends on the fact that the shear and the expansion rate enter into the integral (4.1) along the null geodesic linearly. In contrast, the shear and the expansion rate enter quadratically into the equations of motion (2.3)–(2.5) for the geometry, so the variations do not cancel in the average, and instead we have the generally non-zero backreaction variable $\mathcal{Q}$.

4.3 The distance

For the angular diameter distance, we can apply similar qualitative arguments to obtain the result [45]

$$H\partial_{\bar{z}}\left[(1+\bar{z})^2 H\partial_{\bar{z}}\bar{D}_A\right] \approx -4\pi G_{\rm N}\langle\rho\rangle\bar{D}_A \ , \tag{4.2}$$

where $\bar{D}_A$ is the dominant part of the angular diameter distance with the corrections to the mean dropped, and the same for the redshift, $1+\bar{z} \equiv a(t)^{-1}$. From the conservation of mass, (2.9), it follows that $\langle\rho\rangle \propto (1+z)^3$. The distance is therefore determined entirely by the average expansion rate $H(z)$ and the normalisation of the density today, i.e. $\Omega_{\rm m0}$. For a general FRW model, $\langle\rho\rangle$ in (4.2) would be replaced by $\rho+p$. So the equation for the mean angular diameter distance in terms of $H(z)$ in a statistically homogeneous and isotropic dust universe (with a slowly evolving distribution) is the same as in the FRW ΛCDM model. If backreaction were to produce exactly the same expansion history as the ΛCDM model, the distance-redshift relation would therefore also be identical. This is the case even though the spatial curvature would be large, as the spatial curvature affects the distances

differently than in the FRW case. Note that in a general spacetime, the luminosity distance is related to the angular diameter distance by $D_L = (1+z)^2 D_A$ [27], so from the theoretical point of view it measures the same thing.

Backreaction is not expected to produce an expansion history identical to the ΛCDM model: if the expansion accelerates strongly, then this is likely to be preceded by extra deceleration, unlike in ΛCDM . Therefore the distances will also be different. However, the backreaction distance-redshift relation will be biased towards the ΛCDM model, compared to a FRW model with the same expansion history as in the backreaction case. The reason is that in the FRW model, the equation for D_A is modified not only by the change in $H(z)$, but also by the change in $\rho + p$. This may help to explain why distance observations prefer a value close to -1 for the effective equation of state.

It has been pointed out that the relation between $D_A(z)$ and $H(z)$ can be used as a general test of FRW models [50]. If we measure the distance and the expansion rate independently, we can check whether they satisfy the FRW relation. If this is not the case, the observations cannot be explained in terms of any four-dimensional FRW model. (An extra-dimensional model where the four-dimensional subspace has the FRW metric would still remain a possibility [51].) This holds independent of the presence of dark energy or modified gravity, because light propagation depends directly on the geometry of spacetime, regardless of the equations of motion which determine it. Similarly, we can test the backreaction conjecture that the change in the expansion rate at small redshift is due to structure formation without having a prediction for how the expansion rate changes, simply by checking whether the measured $D_A(z)$ and $H(z)$ satisfy (4.2). The relation (4.2), which violates the FRW consistency condition between expansion and distance is a unique prediction of backreaction which distinguishes it from FRW models. However, the relation between the expansion rate and the distance should be derived more rigorously, and the expected magnitude of the violation is unclear.

The redshift, as well as null geodesic shear and deflection [26], should also be studied in more detail. In particular, it would be interesting to check quantitatively the conjecture that light propagation in a statistically homogeneous and isotropic space with a slowly evolving distribution of small structures can be described in terms of the average expansion rate, and to characterise the small corrections [26, 37, 45]. The small-scale pattern depends only on the angular diameter distance [1], but the effects on large angular scales remain to be determined. Extending to analysis of weak lensing in the case when the geometry is not nearly FRW is also needed for comparing with present and upcoming data. Swiss Cheese models [52], in particular ones with a random distribution of structures [53], are particularly interesting for numerical work, since the average expansion rate and density can be different from the FRW case, and quantities related to light can be explicitly calculated.

It has been argued that deviations from the approximation of treating the matter

as dust would be important for modelling observations of light because of their effect on the way clocks run in different regions of space [54]. Note that the dust approximation does not concern the issue of granularity, or what should be seen as the grains of dust, nor any fundamental aspect of general relativity. It is simply a question of the pressure, anisotropic stress and energy flux being subdominant to the energy density. It seems unlikely that for a matter content of Standard Model particles and cold or warm dark matter these quantities would be so important in a significant fraction of space as to have a major impact on light propagation over large scales. And if that were the case, it is unlikely that the effects would be captured by simply having clocks run at different rates in different regions [26].

5. Discussion

5.1 Beyond Newton

A model is often understood better when it is considered in a larger context, outside its domain of validity. In particular, some special features of FRW models are better appreciated when they are viewed as a limit of general spacetimes with no exact symmetries. One example is the consistency condition between distance and expansion rate discussed above, which is properly viewed as prediction of the FRW model to be observationally tested rather than a fundamental relation. Another aspect is the Newtonian limit of general relativity – or more properly, the relation between Newtonian gravity and general relativity.

Quantifying backreaction analytically or via an improved statistical model similar to the one discussed in section 3.2 is difficult because structure formation is by definition a non-linear process. However, the details of the evolution of non-linear structures starting from small perturbations in the linear regime are routinely studied numerically in cosmological N-body simulations. The problem is that the simulations use Newtonian gravity with periodic boundary conditions. In Newtonian gravity, the variance and the shear cancel in the backreaction variable $\mathcal{Q}$ given in (2.10), up to total derivatives which can be written as boundary terms [55]. Boundary terms of course vanish for periodic boundary conditions. However, using a large simulation and considering boxes of the size of the observable universe would not help the situation. Total derivative terms represent a flux, and due to statistical homogeneity and isotropy, the integrated flux over the boundary should vanish (up to statistical fluctuations), as otherwise there would be a preferred direction.

In general relativity, the backreaction variable $\mathcal{Q}$ does not reduce to a boundary term, and the average expansion rate of a volume depends on the behaviour everywhere in the volume, not just on the boundary. In contrast, the Newtonian evolution is sensitive to boundary conditions, even for infinitely far away boundaries. This is related to the fact that the Poisson equation is elliptic and not hyperbolic, so the

Newtonian system of equations not have a well-posed initial value problem. This is one aspect of the qualitative difference between general relativity and what is called Newtonian cosmology. The small-velocity, weak field limit of general relativity is not Newtonian gravity, as demonstrated by the existence of Newtonian solutions which are not the limit of any general relativity solution [27, 56]. Rather, it is a theory with new degrees of freedom and additional constraints compared to Newtonian gravity [27, 57–60]. The formulation of this limit of general relativity in the cosmological setting with non-linear perturbations is an open issue.

In Newtonian gravity, the feature that inhomogeneities do not change the average expansion rate in a statistically homogeneous and isotropic universe can be understood in terms of energy conservation. In the exactly homogeneous and isotropic case, the Newtonian Friedmann equation (multiplied by a^2) can be interpreted as stating that the kinetic energy plus the potential energy is constant. The relativistic Friedmann equation is mathematically identical, but has a different physical interpretation, with the constant energy replaced by the spatial curvature term. This correspondence does not hold beyond the FRW case. In Newtonian gravity, the total energy is conserved even when the system is inhomogeneous and anisotropic, as long as the system is isolated (i.e. the boundary terms in $\mathcal{Q}$ vanish). However, in general relativity, there is no conservation law for the average spatial curvature, and $a^2\langle{}^{(3)}R\rangle$ is in general not constant. The FRW model is rather special in that the relativistic spatial curvature behaves exactly like the Newtonian energy.

In building a statistical model to evaluate backreaction effects to improve on the semi-realistic treatment discussed in section 4, it is important to make sure that it is consistent with the relativistic evolution equations and constraints, instead of the Newtonian ones. (Similarly, for numerical studies, one should include the relevant relativistic degrees of freedom in the simulation.) For example, if the peak model of section 4 were to be considered in a Newtonian setting, we would have to take into account that the peak identification process does not conserve the Newtonian energy (or correspondingly the relativistic spatial curvature). Taking this constraint into account would completely cancel the effect seen in the model.

While there is no such exact cancellation in general relativity (in the non-linear regime; we discuss the linear case below), it has been argued that for solutions relevant for the real universe there is nevertheless a strong cancellation between the variance of the expansion rate and the shear in the backreaction variable $\mathcal{Q}$ given in (2.10) [61–63]. However, the models which describe a single spherically symmetric structure are not realistic. They only show that in some models backreaction is small, just as it has been demonstrated that it is large for other spherical solutions [34–36]. In [63], a Swiss Cheese construction was also considered. However, the individual holes either have zero shear everywhere except at an infinitely thin shell (where both the shear and the expansion rate diverge), or consist of two regions, one of which has zero variance and zero shear, and the other has zero variance but non-zero

shear. Both cases are unrealistic, the first case because the shear and the expansion rate should remain bounded, and the second because in general non-zero shear is accompanied by non-zero variance of the expansion rate.

5.2 Beyond linearity

It has been argued that backreaction is small, because metric perturbations remain much smaller than unity even when the density perturbation becomes non-linear. However, it is not clear whether metric perturbations indeed remain small. Furthermore, the observables depend not only on the metric but also on its derivatives, which have non-linear fluctuations. See [60] for discussion. A recent paper making this argument is [64], but there the averages are taken over the background space, not the physical volume, so they commute with time derivatives and backreaction is suppressed by construction. (This also means that the results depend on the chosen coordinate system and gauge.) A much more interesting analysis is [65], where the background and perturbations are carefully defined; the approach deserves further study. Also, whether the metric can be written in the perturbed FRW form if backreaction is important is not yet clear, and should be considered.

It has also been suggested that the effect of backreaction could be encapsulated in a change of the evolution of the FRW scale factor. The idea is that backreaction is simply a matter of taking into account the effect of structures on the choice of a FRW background. It can be unambiguously said that this is not the case. If backreaction is important, the universe cannot described by the FRW metric. For example, the relation between the distance and the expansion rate discussed in section 4 which follows directly from the FRW metric is in general violated [26].

Ultimately, the relevant question is not in which form the metric can be written, but what happens to physical quantities. As noted earlier, in the real universe the variation in the local expansion rate is of the same size as the observed change in the average expansion rate, and any realistic metric has to reproduce this fact. The key issue is how the slow and fast expanding regions add up and whether the variations cancel in the average. In linear theory, and in Newtonian gravity, the cancellation holds, but this is not true of non-linear general relativity.

6. Summary

The formation of non-linear structures at late times affects the expansion of the universe and light propagation. This may explain the observed late-time failure of the predictions of homogeneous and isotropic models with ordinary matter and gravity.

Clumpiness can lead to accelerated expansion [11, 32, 34–36]. The observed timescale of 10 billion years and the right order of magnitude for the change of the expansion rate emerge from the known physics of structure formation in a semi-realistic

model [37,38] (though the model does not have acceleration, only less deceleration). However, the model cannot be trusted beyond an order of magnitude, and it is possible that a more detailed study will reveal cancellations which suppress the effect. The physical explanation is simple: at late times, the universe becomes dominated by underdense voids, since they expand more rapidly than their surroundings, so the average expansion rate rises.

The basics of the change in light propagation due to structures and how it is related to the average expansion rate is understood [26, 45]. Demanding that the average quantities give an approximate description of light propagation also fixes the hypersurface of averaging as the one of statistical homogeneity and isotropy. The redshift and the average expansion rate are related in the same way as in FRW models, provided the distribution of structures is statistically homogeneous and isotropic and evolves slowly. In contrast, the relation between the average expansion rate and the angular diameter distance is different from the FRW case. This is a unique prediction which makes it possible to distinguish backreaction from FRW models with dark energy or modified gravity. Details of light propagation remain an important venue for investigation. Interesting issues include understanding the large angle CMB anisotropy and weak lensing in a setting where the spacetime is not assumed to be close to FRW.

An important topic which remains to be properly addressed is the role of non-Newtonian aspects of general relativity [60]. In Newtonian cosmology, backreaction reduces to a boundary term, and is therefore suppressed for a statistically homogeneous and isotropic distribution [55]. However, this is not the case in general relativity, and deriving the cosmological limit of general relativity in the case when the metric is not close to FRW is an open problem. Making sure that the relevant non-Newtonian aspects of general relativity are taken into account is a central issue in going from a semi-realistic model of backreaction to a fully reliable treatment.

There has been much progress in understanding backreaction during the last dozen years. The backreaction conjecture that the failure of the homogeneous and isotropic models with ordinary gravity and matter is due to the known breakdown of homogeneity and isotropy related to structure formation remains a plausible possibility. Directions for further study are clear, and a lot of work remains to be done before we know whether the conjecture is true or false, and if it is true, how to precisely quantify the effect. Until this question has been answered, we do not know whether new physics is needed to explain the observations, or if they can be understood in terms of a complex realisation of general relativity.

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